



Meshless Methods

Soheil Mohammadi

Professor of Computational Mechanics

School of Civil Engineering

University of Tehran

Tel: +98-21-6111 2258, Fax: +98-21-6640 3808, Cell: +98-912-5030 392

Email: smoham@ut.ac.ir, <http://eng.ut.ac.ir/civ/smohammadi>

Nodal Integration for EFG:

- First proposed by Beissel/Belytschko – 1996.
- The strong form of linear elastostatics:

$$\begin{cases} \vec{\nabla} \sigma + b = 0 & \text{in } \Omega \\ u = \bar{u} & \text{on } \Gamma_u \\ \sigma \cdot n = \bar{t} & \text{on } \Gamma_t \end{cases}$$

- The total potential energy of the system can be written as:

$$\Pi(u, \lambda) = \int_{\Omega} \frac{1}{2} \varepsilon : \sigma \, d\Omega - \int_{\Omega} u \cdot b \, d\Omega - \int_{\Gamma_t} u \cdot \bar{t} \, d\Gamma + \int_{\Gamma_u} \lambda \cdot (u - \bar{u}) \, d\Gamma$$

$$\begin{cases} u & \text{an arbitrary } C^0 \text{ function on } \Omega \\ \lambda & \text{a } C^{-1} \text{ function on } \Gamma_u \end{cases}$$

Nodal Integration for EFG:

- This formulation is not stable when the integrals of the variational form are evaluated by a scheme which samples the integrands only at nodes.

$$\delta\Pi = \int_{\Omega} \delta\varepsilon\sigma \, d\Omega - \int_{\Omega} \delta ub \, d\Omega - \int_{\Gamma_t} \delta u \bar{t} \, d\Gamma + \int_{\Gamma_u} \delta\lambda.(u - \bar{u}) \, d\Gamma + \int_{\Gamma_u} \lambda.\delta u \, d\Gamma$$

Nodal Integration for EFG:

- In this formulation, the instability can be eliminated by adding a stabilization term to the functional.
- The stabilization term consists of the square of the residual of the equilibrium equation:

$$\Pi_s(u, \lambda) = \Pi(u, \lambda) + \frac{\alpha_s l_c^2}{E} \int_{\Omega} \left(\vec{\nabla} \cdot \sigma(u) + b \right)^2 d\Omega$$

α_s Dimensionless stabilization parameter

l_c A characteristic length scale of the discretization or nodal arrangement, e.g. average distance to the nearest node.

E Young's modulus

Nodal Integration for EFG:

- Equivalent variational form:

$$\Pi_s(u, \lambda) = \Pi(u, \lambda) + \frac{\alpha_s l_c^2}{E} \int_{\Omega} \left(\vec{\nabla} \cdot \sigma(u) + b \right)^2 d\Omega$$

$$\begin{aligned} \delta \Pi_s(u, \delta u, \lambda, \delta \lambda) = & \int_{\Omega} \delta \varepsilon \sigma(u) d\Omega - \int_{\Omega} \delta u b d\Omega - \int_{\Gamma_t} \delta u \bar{t} d\Gamma \\ & + \int_{\Gamma_u} \delta \lambda \cdot (u - \bar{u}) d\Gamma + \int_{\Gamma_u} \lambda \cdot \delta u d\Gamma \\ & + \frac{2\alpha_s l_c^2}{E} \int_{\Omega} \left(\vec{\nabla} \cdot \delta \sigma \right) \left(\vec{\nabla} \cdot \sigma(u) + b \right) d\Omega \end{aligned}$$

Nodal Integration for EFG:

- The instability is manifested by spurious short wavelength modes.
 - An oscillatory mode with a wavelength of two times the nodal spacing is nearly singular, because its first derivatives almost vanish at the nodes.
 - $\int_{\Omega} \delta \varepsilon \sigma(u) d\Omega$ consists of first derivatives, however, nodal sampling does not adequately reflect the energy of the short wavelength mode, and its contribution to the stiffness matrix is severely underestimated.
 - As a result, the eigenvalue of the strain energy stiffness matrix which corresponds to the short wavelength mode is nearly zero.

Nodal Integration for EFG:

- The stabilization term

$$\frac{2\alpha_s l_c^2}{E} \int_{\Omega} (\vec{\nabla} \cdot \delta \sigma) (\vec{\nabla} \cdot \sigma(u) + b) d\Omega$$

include second derivatives of the displacement. The second derivatives of the short-wavelength modes are not small at the nodes, and therefore, nodal integration of the stabilization term does not underestimate this mode.

- The minimum value of the stabilization term occurs for the same function that gives a stationary value of $\Pi(u, \lambda)$.
- The stabilization term, is a weighted residual of the governing equation.
- Π and Π_s represent a hybrid formulation designed to meet the needs of nodal integration.

Nodal Integration for EFG:

- The trial and test functions are expanded in the MLS basis,

$$u^h(x) = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \Phi(x)d \quad \delta u^h = \Phi(x)\delta d$$

$$\lambda^h(x) = \begin{Bmatrix} \lambda_x \\ \lambda_y \end{Bmatrix} = N(x)\Lambda \quad \delta \lambda^h = N(x)\delta \Lambda$$

- Nodal components (not displacements) d and Lagrange multiplier vector Λ

$$d^T = \{d_{1x} \quad d_{1y} \quad d_{2x} \quad d_{2y} \quad \dots \quad d_{nx} \quad d_{ny}\}$$

$$\Lambda^T = \{\lambda_{1x} \quad \lambda_{1y} \quad \lambda_{2x} \quad \lambda_{2y} \quad \dots \quad \lambda_{qx} \quad \lambda_{qy}\}$$

- Shape functions:

$$\Phi = [\phi_1 \quad \phi_2 \quad \dots \quad \phi_n]$$

$$N = [N_1 \quad N_2 \quad \dots \quad N_q]$$

Nodal Integration for EFG:

- For 2D continuum,

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}$$

$$\boldsymbol{\sigma} = C\boldsymbol{\varepsilon} = CBd$$

$$B = [B_1 \quad B_2 \quad \dots \quad B_n] \quad B_I = \begin{bmatrix} \phi_{I,x} & 0 \\ 0 & \phi_{I,y} \\ \phi_{I,y} & \phi_{I,x} \end{bmatrix}$$

$$b = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \quad \bar{t} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} \quad \bar{u} = \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix}$$

Nodal Integration for EFG:

$$\vec{\nabla} \cdot \sigma = \begin{Bmatrix} \sigma_{xx,x} + \sigma_{yx,x} \\ \sigma_{yy,y} + \sigma_{xy,y} \end{Bmatrix} = C' D d$$

$$D = [D_1 \quad D_2 \quad \dots \quad D_n]$$

$$D_I = \phi_{I,x} \begin{bmatrix} \phi_{I,xx} & 0 \\ \phi_{I,xy} & 0 \\ \phi_{I,yy} & 0 \\ 0 & \phi_{I,xx} \\ 0 & \phi_{I,xy} \\ 0 & \phi_{I,yy} \end{bmatrix}$$

Nodal Integration for EFG:

	C	C'
plane stress	$\frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{1}{1-\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\frac{E}{1+\nu} \begin{bmatrix} \frac{1}{1-\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\nu}{1-\nu} & 0 \\ \frac{1}{1-\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\nu}{1-\nu} & 0 \\ 0 & \frac{\nu}{1-\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{1-\nu} \\ 0 & \frac{\nu}{1-\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{1-\nu} \end{bmatrix}$
plane strain	$\frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{\nu}{1-2\nu} & 0 \\ \frac{\nu}{1-2\nu} & \frac{1-\nu}{1-2\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\frac{E}{1+\nu} \begin{bmatrix} \frac{1-\nu}{1-2\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\nu}{1-2\nu} & 0 \\ \frac{1-\nu}{1-2\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{\nu}{1-2\nu} & 0 \\ 0 & \frac{\nu}{1-2\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1-\nu}{1-2\nu} \\ 0 & \frac{\nu}{1-2\nu} & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1-\nu}{1-2\nu} \end{bmatrix}$

Nodal Integration for EFG:

- The variational weak form then becomes,

$$\begin{aligned} & \delta d^T \int_{\Omega} (B^T C B d - \Phi^T b) d\Omega - \delta d^T \int_{\Gamma_t} \Phi^T \bar{t} d\Gamma + \\ & \delta \Lambda^T \int_{\Gamma_u} N^T (\Phi^T d - \bar{u}) d\Gamma + \delta d^T \int_{\Gamma_u} \Phi^T N \Lambda d\Gamma + \\ & \frac{2\alpha_s l_c^2}{E} \delta d^T \int_{\Omega} (D^T C'^T C' D + D^T C'^T b) d\Omega = 0 \end{aligned}$$

- For arbitrary δd , $\delta \Lambda$:

$$\begin{bmatrix} K + K_s & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} d \\ \Lambda \end{bmatrix} = \begin{bmatrix} f_{ext} + f_s \\ f_{\lambda} \end{bmatrix}$$

Nodal Integration for EFG:

$$\begin{bmatrix} K + K_s & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} d \\ \Lambda \end{bmatrix} = \begin{bmatrix} f_{ext} + f_s \\ f_\lambda \end{bmatrix}$$

$$K = \int_{\Omega} B^T C B d\Omega$$

$$K_s = \frac{2\alpha_s l_c^2}{E} \delta d^T \int_{\Omega} D^T C'^T C' D d\Omega$$

$$G = \int_{\Gamma_u} \Phi^T N d\Gamma$$

$$f_{ext} = \int_{\Omega} \Phi^T b d\Omega + \int_{\Gamma_t} \Phi^T \bar{t} d\Gamma$$

$$f_s = -\frac{2\alpha_s l_c^2}{E} \int_{\Omega} D^T C'^T b d\Omega$$

$$f_\lambda = \int_{\Gamma_u} N^T \bar{u} d\Gamma$$

Nodal Integration for EFG:

- By sampling the integrands only at nodal points:

$$K \approx \sum_{I=1}^n q_I^{\Omega} B^T(x_I) C(x_I) B(x_I)$$

$$K_s \approx \frac{2\alpha_s l_c^2}{E} \sum_{I=1}^n q_I^{\Omega} D^T(x_I) C'^T(x_I) C'(x_I) D(x_I)$$

$$G \approx \sum_{I=1}^r q_I^{\Gamma_u} \Phi^T(x_I) N(x_I)$$

$$f_{ext} \approx \sum_{I=1}^n q_I^{\Omega} \Phi^T(x_I) b(x_I) + \sum_{I=1}^s q_I^{\Gamma_t} \Phi^T(x_I) \bar{t}(x_I)$$

$$f_s \approx -\frac{2\alpha_s l_c^2}{E} \sum_{I=1}^n q_I^{\Omega} D^T(x_I) C'^T(x_I) b(x_I)$$

$$f_{\lambda} \approx \sum_{I=1}^r q_I^{\Gamma_u} N^T(x_I) \bar{u}(x_I)$$

Nodal Integration for EFG:

$$q_I^\Omega = \frac{f_{\Omega,I} \rho_I^2}{\sum_{J=1}^n f_{\Omega,J} \rho_J^2} A_\Omega \quad (\text{the fraction of the total area represented by node } I)$$

$$q_I^{\Gamma_u} = \frac{f_{\Gamma_u,I} \rho_I}{\sum_{J=1}^r f_{\Gamma_u,J} \rho_J} L_{\Gamma_u}$$

n = number of nodes in Ω
 r = number of nodes on Γ_u
 s = number of nodes on Γ_t

$$q_I^{\Gamma_t} = \frac{f_{\Gamma_t,I} \rho_I}{\sum_{J=1}^s f_{\Gamma_t,J} \rho_J} L_{\Gamma_t}$$

Nodal Integration for EFG:

A_{Ω} = the total area

ρ_I = the radius of the circular support of the weighting function associated with node I

L_{Γ_u} = the total length of the essential boundary

L_{Γ_t} = the total length of the natural boundary

$f_{\Omega,I}$ = the fraction of the support which lies in Ω $\left\{ \begin{array}{ll} 1 & : \text{interior nodes} \\ 0.50 & : \text{nodes on a straight edge} \\ 0.25 & : \text{nodes on right corner} \end{array} \right.$

$f_{\Gamma_u,I}$ = the fraction of $2\rho_I$ the essential boundary covers in the support of the weight function W_I

$f_{\Gamma_t,I}$ = the fraction of $2\rho_I$ the natural boundary covers in the support of the weight function W_I

Nodal Integration for EFG:

- Alternative formulae for q_I^Ω
- This alternative may be more suitable for large meshes
- The components are evaluated from a mass-like matrix (row-sum scheme)

$$q_I^\Omega = \sum_J M'_{IJ}$$

$$M'_{IJ} = \int_{\Omega} \phi_I(x) \phi_J(x) d\Omega$$

- M'_{IJ} is evaluated by using a domain quadrature scheme such as cells in EFG.
- The method ensures the conservation of the total area (volume) of the domain:

$$\sum_I q_I^\Omega = \Omega$$

Nodal Integration for EFG:

1. The full Gaussian integration is more accurate and displays a higher rate of convergence than nodal integration.
2. The accuracy of stabilized nodal integration is significantly lower than the complete Gaussian integration.

“the price of the increased speed and simplicity”

3. As α increases, the error converges to the value associated with the least square term, which is less accurate than the Galerkin formulation.
4. The un-stabilized equations are not singular, and they are invertible. However, they are nearly singular and may produce oscillatory solutions.

Nodal Integration for EFG:

5. Due to its approximate nature, it may be proposed to use an approximation to the calculations of the derivatives of the shape functions:
 1. By dropping the derivatives of A^{-1} and B in the expansion for the derivatives of the shape functions (similar to the Diffuse Element Method – DEM).
 2. The method ignores the dependence of $a(x)$ on x ; ignoring the ‘moving’ part of the MLS shape functions.
 3. The approximation can substantially save time in explicit dynamic computations.
 4. The additional errors are significant, and ignoring $a_{,I}$ and $a_{,IJ}$ is not recommended.

Nodal Integration for EFG:

6. The stabilization procedure may decrease accuracy if the original solution does not contain the unstable mode.
7. Effective extension of the stabilization method to nonlinear dynamics still requires some substantive developments.

